Analytic solutions to the shallow water equations

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Analytic, two-dimensional steady-state solutions to the rotating shallow water equations over variable topography are derived, by exploiting a drastic simplification of the equilibrium problem that occurs for nondivergent flows. For such flows, the equilibrium system decouples, and the cross-stream component of the momentum equation formally reduces to the barotropic vorticity equation. Thus, any solution to the barotropic vorticity equation can be used, in principle, to construct exact equilibrium solutions to the shallow water equations.

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Exact solutions to the basic equations of fluid mechanics are precious, both for their intrinsic value and for their potential use in the validation of numerical methods. In the present work, we derive solutions to the shallow water equations (SWE), a reduced-dimension framework widely used for the investigation of atmosphere and ocean dynamics, as well as for engineering flow problems. Very few exact solutions to the SWE have been obtained to date, most of them describing simple flows in idealized environments. Examples are the one-dimensional (1D) solutions in rotating parabolic channels described by Shapiro [1], to which we also refer for a synopsis of the classical literature in the field, and the two-dimensional (2D) steady-state solutions in a paraboloidal basin by Ball [2]. Due to the complex mathematical structure of the equilibrium system, only nondivergent solutions have been obtained to date. This represents an important limitation from the point of view of the applications that nobody has yet been able to overcome.

Here we present a new method for constructing nondivergent, 2D equilibrium solutions over variable topography, that makes use of a nonconventional formulation of the SWE equilibrium problem. In this formulation, the existence of a functional relationship between the equilibrium Bernoulli function B and potential vorticity (PV) q [$q=q(\psi)=dB(\psi)/d\psi$, where ψ is the stream function of the momentum field] is exploited to write the rotating, inviscid equilibrium system as a set of two coupled pde's for ψ and the fluid depth h, depending on the free function $B(\psi)$. This (ψ,h) formulation has appeared several times in the literature (see, among others, Refs. [3–6]), but, to the best of our knowledge, has never been used to derive 2D solutions of the SWE.

We start with a brief derivation of the (ψ,h) system from the familiar equilibrium equations:

$$(\zeta+f)\mathbf{k}\times\mathbf{v}+\nabla B=\mathbf{0}, \tag{1}$$

$$\nabla\cdot(h\mathbf{v})=0, \tag{2}$$

where h is the depth of the fluid layer, i.e., the distance between the free surface and the varying height h_b of the bottom topography, $\mathbf{v}$ is the two-dimensional flow velocity, f the Coriolis parameter, and $\mathbf{k}$ the unit vector in the outward direction. The relative vorticity ζ, and the Bernoulli function B, are given by

$$\zeta=\mathbf{k}\cdot\nabla\times\mathbf{v};\quad B=v^2/2+gh+gh_b, \tag{3}$$

where v is the magnitude of the flow, and g the acceleration due to gravity. Equation (2) can be satisfied by introducing a "transport" stream function ψ:

$$h\mathbf{v}=\mathbf{k}\times\nabla\psi. \tag{4}$$

It follows from the component of Eq. (1) along $\mathbf{v}$ that $B=B(\psi)$, while the $\nabla\psi$ component of the same equation gives a simple relation between q and the Bernoulli function

$$q\equiv\frac{\zeta+f}{h}=q(\psi)=\frac{dB}{d\psi}. \tag{5}$$

Using Eqs. (3)–(5), we can finally write the equilibrium problem as a set of two coupled pde's for ψ and h:

$$\nabla\cdot\left(\frac{1}{h}\nabla\psi\right)=-f+h\frac{dB}{d\psi}, \tag{6}$$

$$\frac{1}{2}\frac{|\nabla\psi|^2}{h^2}+gh+gh_b=B(\psi), \tag{7}$$

depending on the free function $B(\psi)$. For given ψ and given topography, Eq. (7) can be considered, at each spatial location, as a cubic, algebraic equation for h. Any positive real root of the cubic corresponds to an expression of the form $h=h(\mathbf{x},\psi,|\nabla\psi|^2)$, that can be replaced in Eq. (6), giving an equation for ψ only. In other words, the system (6) and (7) can be viewed as a single second order, quasilinear pde for ψ, coupled with an algebraic constraint.

Let us now consider nondivergent solutions of the system (6) and (7). For a nondivergent flow defined by a streamfunction ξ,

$$\mathbf{v}=\mathbf{k}\times\nabla\xi, \tag{8}$$

Eq. (4) implies that ψ and h are functions of ξ, and that

$$h(\xi)=\frac{d\psi}{d\xi}. \tag{9}$$

As a consequence, the system (6) and (7) becomes

$$\nabla^2 \xi = -f + \frac{dB}{d\xi}, \tag{10}$$

$$\frac{1}{2}|\nabla \xi|^2 = B(\xi) - gh(\xi) - gh_b. \tag{11}$$

Clearly, a crucial simplification has occurred, since the equilibrium system is now decoupled. Once B is specified as a function of ξ, Eq. (10) formally reduces to the barotropic vorticity equation (BVE); if we can solve this equation for ξ, the Bernoulli constraint (11) simply becomes a definition of the surface elevation $h+h_b$.

Thus, any analytic solution to the BVE could be used, in principle, to derive exact shallow water equilibria. We have reversed here the usual approach to the problem: instead of trying to solve the SWE equilibrium equations for some given (simple) topography, we prescribe some (simple) functional dependence of B on ξ, such that we can find solutions to the BVE, and then ask what topography would make such solutions exact solutions to the SWE. This procedure cannot define univocally the bottom topography, since there is no *a priori* prescription on how the surface elevation should be split among h and h_b: the only constraint is that h needs to be a function of ξ. Thus, the same flow can generate an infinity of solutions of the SWE, corresponding to an infinity of different topographies. In order to remove this degeneracy, we shall choose topography in such a way as to yield a linear $h(\xi)$, which in turn gives a quadratic $\psi(\xi)$.

We now discuss some explicit, constant-f, 2D solutions to the system (10) and (11).

(i) *Solutions over constant topography*. If h_b is constant, we must find a stream function $\xi=\xi(x,y)$ such that $\nabla^2\xi$ and $|\nabla\xi|^2$ are both functions of ξ. A trivial solution is given by unidirectional flows with straight streamlines, since in this case ξ depends on a linear combination of x and y. Next, we consider flows with a quadratic stream function,

$$\xi = \frac{a}{2}x^2 + cxy + \frac{b}{2}y^2. \tag{12}$$

Such flows have velocity components that vary linearly with x and y, and constant vorticity

$$\nabla^2 \xi = a + b \equiv \zeta_0. \tag{13}$$

It follows from Eq. (12) that

$$|\nabla \xi|^2 = (a^2+c^2)x^2 + 2c(a+b)xy + (b^2+c^2)y^2. \tag{14}$$

Unfortunately, the expression on the rhs of Eq. (14) is proportional to ξ only in two trivial cases: either when $a=b$, $c=0$, which corresponds to circular streamlines, or when $c^2=ab$, which again gives straight streamlines. Thus, elliptic vortices cannot be solution to the system (10) and (11) in absence of a varying topography. In fact, this conclusion can be made more general by noticing that Eq. (12) can be identified with the Taylor expansion, up to second order, of a generic streamfunction with an extremum at the origin. This implies that no solution to the system (10) and (11) with closed, noncircular streamlines can be found in absence of a variable topography.

(ii) *Linear* $B(\xi)$: *the Ball solutions*. Let us further examine the case of flows with elliptic streamlines and constant vorticity. Without loss of generality, we can take $c=0$ in Eq. (12), restricting to flows with streamlines elongated either in the x or in the y direction. Equation (11) indicates that such equilibria could be sustained by a topography with quadratic dependences on x and y. Indeed, steady, and even time-dependent solutions in channels of parabolic cross section or in paraboloidal basins have long been known (see [1], and references therein). Constant vorticity solutions for flows in a basin with the shape of an elliptic paraboloid,

$$h_b = \frac{\alpha}{2}x^2 + \frac{\beta}{2}y^2, \quad \alpha,\beta > 0, \tag{15}$$

were given by Ball [2], and have been discussed more recently in Ref. [7]. In the following, we give a brief alternative derivation of these solutions, based on the system (10) and (11). Clearly, constant vorticity implies a linear $B(\xi)$,

$$B = B_0 + (\zeta_0 + f)\xi, \tag{16}$$

with B_0 an arbitrary constant. Then, since B, h_b, and $|\nabla\xi|^2$ are all quadratic functions of the coordinates, the Bernoulli relation forces h to be a linear function of ξ,

$$h = h_0 + h_1\xi. \tag{17}$$

Substituting Eqs. (15)–(17) in Eq. (11), and equating to zero the coefficients of the different powers of x and y, yields $B_0=gh_0$ and

$$a^2 - (\zeta_0 + f - gh_1)a + g\alpha = 0, \tag{18}$$

$$b^2 - (\zeta_0 + f - gh_1)b + g\beta = 0. \tag{19}$$

The latter equations show that solutions with circular streamlines ($a=b$) are only possible if the basin has the shape of a circular paraboloid. Equations (18) and (19), together with Eq. (13), can be solved to express a, b, and h_1 in terms of the free parameters, i.e., the vorticity ζ_0 and the coefficients (α,β), determining the shape of the vessel. In particular, Eqs. (18) and (19) can be combined to give a cubic equation for an elongation parameter $Y\equiv a-b$,

$$Y^3 - [\zeta_0^2 - 2g(\alpha+\beta)]Y - 2g\zeta_0(\alpha-\beta) = 0. \tag{20}$$

The corresponding solution for h_1 is

$$h_1 = \frac{f}{g} - \frac{\alpha-\beta}{Y} \quad (Y \neq 0),$$

$$h_1 = \frac{f}{g} + \frac{\zeta_0}{2g} - \frac{2\alpha}{\zeta_0} \quad (Y=0). \tag{21}$$

The cubic (20) could be obtained from Equations (7.12), (7.16), and (7.17) of Ref. [2], by eliminating the variables r and s in favor of M, which corresponds to our Y. A different elimination procedure was instead suggested in [2], and carried out in [7], which leads to more complicated equations, that are only amenable to numerical investigation.

The possible existence of multiple equilibria for a given basin shape is apparent from Eq. (20): in the limit $\alpha=\beta$, we find a root $Y=0$, that corresponds to flows with circular streamlines, and, for large enough vorticity ($\zeta_0^2 > 4g\alpha$), two additional roots $Y = \pm\sqrt{\zeta_0^2 - 4g\alpha}$, corresponding to flows with elliptic streamlines, elongated either in the x or in the y direction. Multiple roots can also be found for $\alpha \neq \beta$. It may be shown, however, that these solutions have very large values of the Rossby number, as originally noted in Ref. [2].

A last feature of the Ball solutions worth noticing is that they can only be realized in basins bounded by vertical walls. There cannot be a shoreline, i.e., a line on which the depth h vanishes, since the PV would diverge when approaching such line, making the solution unphysical. One could hope to remove this limitation by modulating the radial dependence of the vorticity field, while keeping streamlines of elliptic shape. Unfortunately, this is not possible. Consider a stream function of the form

$$\xi = \xi(\phi), \quad \phi = \frac{a}{2}x^2 + \frac{b}{2}y^2. \tag{22}$$

Replacing this ansatz in Eq. (10), we obtain

$$\dot{\xi}\nabla^2\phi + \ddot{\xi}|\nabla\phi|^2 = -f + \frac{dB}{d\xi}, \tag{23}$$

where the dot indicates derivative with respect to ϕ. If $a \neq b$, this equation can only be satisfied if the second term on the lhs vanishes identically, i.e., if $\ddot{\xi}=0$ everywhere. This implies a linear $\xi(\phi)$, and we recover the Ball solutions. Therefore, the Ball solutions are the only nondivergent equilibrium solutions of the SWE on the f-plane with elliptic streamlines.

(iii) *Quadratic* $B(\xi)$. Equilibria with a shoreline, and finite PV, can be obtained by choosing a quadratic functional dependence of B on ξ. This gives a linear BVE, that has some well-known elementary solutions. For example, if

$$\xi = A\cosh(x/L) + C\cosh(y/L), \tag{24}$$

where L is a length scale, and A, C free coefficients, the relative vorticity is given by

$$\nabla^2\xi = (1/L^2)\xi, \tag{25}$$

and the flow (24) is a solution of Eq. (10) if

$$B = B_0 + f\xi + \frac{1}{2L^2}\xi^2. \tag{26}$$

When the constants A and C have the same sign, the flow described by Eq. (24) has closed streamlines, with quasielliptical shapes for small values of x and y. It follows from Eq. (24) that

$$L^2|\nabla\xi|^2 = \xi^2 - A^2 - C^2 - 2AC\cosh(x/L)\cosh(y/L). \tag{27}$$

Consequently, the quadratic terms in $|\nabla\xi|^2/2$ and B exactly cancel in the Bernoulli relation. Choosing a linear $h(\xi)$, as in Eq. (17), allows us to balance the linear term in $B(\xi)$, and we obtain equilibrium solutions if the bottom topography has the following shape:

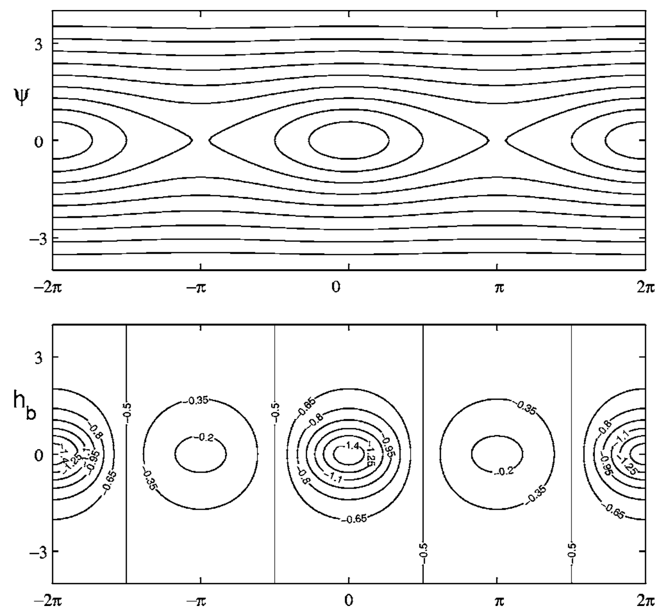

FIG. 1. Top panel: stream function of Suart's (cat's eyes) vortex row solution of the barotropic vorticity equation. This becomes an exact, steady-state solution of the SWE, in the presence of the bottom topography whose contour levels are shown in the lower panel.

$$h_b = H\cosh(x/L)\cosh(y/L), \quad H > 0. \tag{28}$$

Substitution of Eqs. (26)–(28) into the Bernoulli relation (11) yields the constraints

$$B_0 = gh_0 - \frac{1}{2L^2}(A^2 + C^2), \tag{29}$$

$$gh_1 = f, \tag{30}$$

$$AC = gHL^2. \tag{31}$$

Note that h_0 is still free, and that Eq. (31) only specifies the product AC. Thus, for a given topography (i.e., given H and L), depending on h_0, and on the relative sizes of A and C, we find a double infinity of possible equilibria. These equilibria can have a shoreline if the coefficients (A, C) and f have opposite signs (i.e., in the case of anticyclonic circulations). Since

$$q = \frac{f + (1/L^2)\xi}{h_0 + (f/g)\xi}, \tag{32}$$

the only way to have finite PV at the shoreline is to have constant PV throught the flow. This requires $L = \sqrt{gh_0}/|f|$, which fixes h_0, once L and f have been specified.

Clearly, Eq. (30) implies geostrophic balance, which could not correspond to a steady solution of the SWE in absence of a varying topography. The gradient of the topography (28) is exactly what is needed to locally balance nonlinear advection, i.e.,

$$\mathbf{v}\cdot\nabla\mathbf{v} = -g\nabla h_b, \tag{33}$$

as it can be easily verified by a direct calculation of the advection term. It may be noted that this equation implies the vanishing of the curl of the advection term, similarly to what happens for generalized Beltrami flows.

(iv) *Stuart's vortex row*. The last example we shall discuss makes use of a well-known solution of the two-dimensional Euler equation given by Stuart [8]. The flow is given by

$$\xi = \xi_0 \log(\cosh y - \epsilon \cos x), \tag{34}$$

with ξ_0 and ϵ costants, and represents an infinite row of identical vortices. It follows from Eq. (34) that

$$\nabla^2 \xi = \xi_0 (1 - \epsilon^2) e^{-2\xi/\xi_0}, \tag{35}$$

showing that the Stuart's vortex row is a solution of Liouville's equation. The stream function (34) is a solution of Eq. (10) for the Bernoulli function

$$B = B_0 + f\xi - \frac{1}{2}\xi_0^2 (1 - \epsilon^2) e^{-2\xi/\xi_0}. \tag{36}$$

On the other hand, it is readily found that

$$|\nabla \xi|^2 = -\xi_0^2 (1 - \epsilon^2) e^{-2\xi/\xi_0} + \xi_0^2 \frac{\cosh y + \epsilon \cos x}{\cosh y - \epsilon \cos x}. \tag{37}$$

Thus, exponential terms cancel out in the Bernoulli relation, $h(\xi)$ can be taken as a linear function of ξ, in order to balance the linear term in $B(\xi)$, and we find a solution of the equilibrium system if the bottom topography has the following form:

$$gh_b = -\frac{\xi_0^2}{2} \frac{\cosh y + \epsilon \cos x}{\cosh y - \epsilon \cos x}. \tag{38}$$

A constant could, of course, be added to this expression to get a positive definite h_b; this would only affect the definition of B_0. Stream function and bottom topography, for $\epsilon = 0.5$ and $\xi_0 = 1$, are plotted in Fig. 1. At small values of y, the topography along x is an alternance of hills and valleys; the vortices sit on topography minima, and this does not depend on the sign of the circulation. At large values of y, the flow tends asymptotically towards a zonal flow, and h_b towards a constant value. Differently from the previous examples, there is no free parameter in this solution; given a topography, the equilibrium flow is univocally determined.

Finally, we notice that the equilibrium system remains coupled if $B(\psi)$ [and not $B(\xi)$] is prescribed. In this case, exact solutions may be obtained for unidirectional flows (examples for cartesian flows have been given in Ref. [6]), but the derivation of exact 2D solutions appears very difficult. In the case of divergent flows, the equilibrium system is even more complex. In this case, the formulation based on Eqs. (6) and (7) might prove useful in the construction of a numerical approach to the problem.

[1] A. Shapiro, J. Fluid Mech. **318**, 49 (1996).
[2] F. K. Ball, J. Fluid Mech. **22**, 529 (1965).
[3] A. E. Gill, J. Fluid Mech. **80**, 641 (1977).
[4] D. Nof, J. Phys. Oceanogr. **11**, 1662 (1981).
[5] P. Ripa, J. Phys. Oceanogr. **30**, 2504 (2000).
[6] P. H. Chavanis and J. Sommeria, Phys. Rev. E **65**, 026302 (2002).
[7] J. S. Allen, J. A. Barth, and P. A. Newberger, J. Phys. Oceanogr. **20**, 1017 (1990).
[8] J. T. Stuart, J. Fluid Mech. **29**, 417 (1967).